Find the exact value of each trigonometric expression.

45. cos 15°

SOLUTION:

Write 15° as the sum or difference of angle measures with cosines that you know.

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

ANSWER:

$$\frac{\sqrt{6}+\sqrt{2}}{4}$$

46. sin 345°

SOLUTION:

Write 345° as the sum or difference of angle measures with sines that you know.

$$\begin{aligned}\sin 345^\circ &= \sin(300^\circ + 45^\circ)\\ &= \sin 300^\circ \cos 45^\circ + \cos 300^\circ \sin 45^\circ\\ &= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= -\frac{\sqrt{6}-\sqrt{2}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

ANSWER:

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

47. $\tan \frac{13\pi}{12}$

SOLUTION:

$$\begin{aligned}
\tan\frac{13\pi}{12} &= \tan\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)\\
&= \frac{\tan\frac{5\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{5\pi}{6}\tan\frac{\pi}{4}}\\
&= \frac{-\frac{\sqrt{3}}{3}+1}{1-\left(-\frac{\sqrt{3}}{3}\right)(1)}\\
&= \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}\\
&= \frac{3-\sqrt{3}}{3+\sqrt{3}}\\
&= \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}\\
&= \frac{9-6\sqrt{3}+3}{9-3}\\
&= \frac{12-6\sqrt{3}}{6}\\
&= 2-\sqrt{3}
\end{aligned}$$

ANSWER:

$2 - \sqrt{3}$

48. $\sin \frac{7\pi}{12}$

SOLUTION:

Write $\frac{7\pi}{12}$ as the sum or difference of angle measures with sines that you know.

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4}+\cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

ANSWER:

$$\frac{\sqrt{6}+\sqrt{2}}{4}$$

49. $\cos -\frac{11\pi}{12}$

SOLUTION:

Write $-\frac{11\pi}{12}$ as the sum or difference of angle measures with cosines that you know.

$$\begin{aligned}\cos-\frac{11\pi}{12} &= \cos\left(-\frac{\pi}{6}-\frac{3\pi}{4}\right)\\ &= \cos-\frac{\pi}{6}\cos\frac{3\pi}{4}+\sin-\frac{\pi}{6}\sin\frac{3\pi}{4}\\ &= \frac{\sqrt{3}}{2}\cdot-\frac{\sqrt{2}}{2}+\left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

ANSWER:

$$\frac{-\sqrt{6}-\sqrt{2}}{4}$$

50. $\tan \frac{5\pi}{12}$

SOLUTION:

Write $\frac{5\pi}{12}$ as the sum or difference of angle measures with tangents that you know.

$$\begin{aligned}\tan\frac{5\pi}{12} &= \tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\\ &= \frac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}\\ &= \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}(1)}\\ &= \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}\\ &= \frac{3+\sqrt{3}}{3-\sqrt{3}}\\ &= \frac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}}\\ &= \frac{12+6\sqrt{3}}{6}\\ &= 2+\sqrt{3}\end{aligned}$$

ANSWER:

$2+\sqrt{3}$

Simplify each expression.

51. $\dfrac{\tan\frac{\pi}{9}+\tan\frac{8\pi}{9}}{1-\tan\frac{\pi}{9}\tan\frac{8\pi}{9}}$

SOLUTION:

$$\begin{aligned}\frac{\tan\frac{\pi}{9}+\tan\frac{8\pi}{9}}{1-\tan\frac{\pi}{9}\tan\frac{8\pi}{9}} &= \tan\left(\frac{\pi}{9}+\frac{8\pi}{9}\right)\\ &= \tan\pi\\ &= 0\end{aligned}$$

ANSWER:

0

52. $\cos 24° \cos 36° - \sin 24° \sin 36°$

SOLUTION:

$$\begin{aligned}\cos 24° \cos 36° - \sin 24° \sin 36° &= \cos(24° + 36°)\\ &= \cos 60°\\ &= \frac{1}{2}\end{aligned}$$

ANSWER:

$\frac{1}{2}$

53. $\sin 95° \cos 50° - \cos 95° \sin 50°$

SOLUTION:

$$\begin{aligned}\sin 95° \cos 50° - \cos 95° \sin 50° &= \sin(95° - 50°)\\ &= \sin 45°\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

ANSWER:

$\frac{\sqrt{2}}{2}$

54. $\cos \frac{2\pi}{9} \cos \frac{\pi}{18} + \sin \frac{2\pi}{9} \sin \frac{\pi}{18}$

SOLUTION:

$$\begin{aligned}\cos \frac{2\pi}{9} \cos \frac{\pi}{18} + \sin \frac{2\pi}{9} \sin \frac{\pi}{18} &= \cos\left(\frac{2\pi}{9} - \frac{\pi}{18}\right)\\ &= \cos \frac{\pi}{6}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

ANSWER:

$\frac{\sqrt{3}}{2}$

Verify each identity.

55. $\cos(\theta + 30°) - \sin(\theta + 60°) = -\sin\theta$

SOLUTION:

$$\begin{aligned}&\cos(\theta + 30°) - \sin(\theta + 60°)\\&= \cos\theta\cos 30° - \sin\theta\sin 30° - (\sin\theta\cos 60° + \cos\theta\sin 60°)\\&= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\\&= -\sin\theta\end{aligned}$$

ANSWER:

$$\begin{aligned}&\cos(\theta + 30°) - \sin(\theta + 60°)\\&= \cos\theta\cos 30° - \sin\theta\sin 30° - (\sin\theta\cos 60° + \cos\theta\sin 60°)\\&= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\\&= -\sin\theta\end{aligned}$$

56. $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta)$

SOLUTION:

$$\begin{aligned}&\cos\left(\theta + \frac{\pi}{4}\right)\\&= \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} && \text{Cosine Sum Identity}\\&= \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta && \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \text{ and } \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\\&= \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta) && \text{Factor } \frac{\sqrt{2}}{2} \text{ from each term.}\end{aligned}$$

ANSWER:

$$\begin{aligned}\cos\left(\theta + \frac{\pi}{4}\right) &= \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}\\&= \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\\&= \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta)\end{aligned}$$

57. $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

SOLUTION:

$$\begin{aligned}
&\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) \\
&= \cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3} + \cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3} \\
&= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \\
&= \cos\theta
\end{aligned}$$

ANSWER:

$$\begin{aligned}
\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) &= \cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3} + \cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3} \\
&= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \\
&= \cos\theta
\end{aligned}$$

58. $\tan\left(\theta + \frac{3\pi}{4}\right) = \frac{\tan\theta - 1}{\tan\theta + 1}$

SOLUTION:

$$\begin{aligned}
&\tan\left(\theta + \frac{3\pi}{4}\right) \\
&= \frac{\tan\theta + \tan\frac{3\pi}{4}}{1 - \tan\theta\tan\frac{3\pi}{4}} && \text{Tangent Sum Identity} \\
&= \frac{\tan\theta - 1}{1 - \tan\theta\cdot(-1)} && \tan\frac{3\pi}{4} = -1 \\
&= \frac{\tan\theta - 1}{\tan\theta + 1} && \text{Simplify.}
\end{aligned}$$

ANSWER:

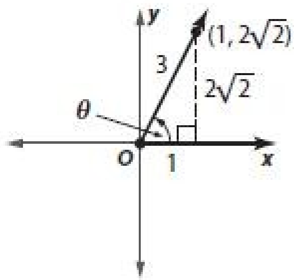
$$\begin{aligned}
\tan\left(\theta + \frac{3\pi}{4}\right) &= \frac{\tan\theta + \tan\frac{3\pi}{4}}{1 - \tan\theta\tan\frac{3\pi}{4}} \\
&= \frac{\tan\theta - 1}{1 - \tan\theta\cdot(-1)} \\
&= \frac{\tan\theta - 1}{\tan\theta + 1}
\end{aligned}$$

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.

59. $\cos\theta = \frac{1}{3}, (0°, 90°)$

SOLUTION:

Since $\cos\theta = \frac{1}{3}$ on the interval (0°, 90°), one point on the terminal side of θ has x-coordinate 1 and a distance of 3 units from the origin as shown. The y-coordinate of this point is therefore $\sqrt{3^2 - 1^2}$ or $2\sqrt{2}$.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $\frac{2\sqrt{2}}{3}$ and $\tan\theta = \frac{y}{x}$ or $2\sqrt{2}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right) \\ &= \frac{4\sqrt{2}}{9}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{3}\right)^2 - 1 \\ &= 2\left(\frac{1}{9}\right) - 1 \\ &= \frac{2}{9} - \frac{9}{9} \\ &= -\frac{7}{9}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2(2\sqrt{2})}{1 - (2\sqrt{2})^2} \\ &= \frac{4\sqrt{2}}{1 - 8} \\ &= -\frac{4\sqrt{2}}{7}\end{aligned}$$

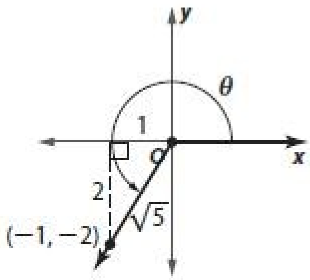
ANSWER:

$\frac{4\sqrt{2}}{9}, -\frac{7}{9}, -\frac{4\sqrt{2}}{7}$

60. tan θ = 2, (180°, 270°)

SOLUTION:

If tan θ = 2, then tan θ = $\frac{2}{1}$. Since tan θ = $\frac{2}{1}$ on the interval (180°, 270°), one point on the terminal side of θ has *x*-coordinate −1 and *y*-coordinate −2 as shown. The distance from the point to the origin is $\sqrt{2^2+1^2}$ or $\sqrt{5}$.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{2\sqrt{5}}{5}$ and $\cos\theta = \frac{x}{r}$ or $-\frac{\sqrt{5}}{5}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{2\sqrt{5}}{5}\right)\left(-\frac{\sqrt{5}}{5}\right)\\ &= \frac{4}{5}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{\sqrt{5}}{5}\right)^2 - 1\\ &= 2\left(\frac{5}{25}\right) - 1\\ &= \frac{10}{25} - \frac{25}{25}\\ &= -\frac{3}{5}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2(2)}{1-(2)^2}\\ &= \frac{4}{1-4}\\ &= -\frac{4}{3}\end{aligned}$$

ANSWER:

$\frac{4}{5}, -\frac{3}{5}, -\frac{4}{3}$

Find the exact value of each expression.

63. $\sin 75°$

SOLUTION:

Notice that 75° is half of 150°. Therefore, apply the half-angle identity for sine, noting that since 75° lies in Quadrant I, its sine is positive.

$$\begin{aligned}\sin 75° &= \sin \frac{150°}{2}\\ &= \sqrt{\frac{1-\cos 150°}{2}}\\ &= \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}\\ &= \sqrt{\frac{2+\sqrt{3}}{4}}\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

ANSWER:

$$\frac{\sqrt{2+\sqrt{3}}}{2}$$

64. $\cos \frac{11\pi}{12}$

SOLUTION:

Notice that $\frac{11\pi}{12}$ is half of $\frac{11\pi}{6}$. Therefore, apply the half-angle identity for cosine, noting that since $\frac{11\pi}{12}$ lies in Quadrant II, its cosine is negative.

$$\begin{aligned}\cos\frac{11\pi}{12} &= \cos\frac{\frac{11\pi}{6}}{2} \\ &= -\sqrt{\frac{1+\cos\frac{11\pi}{6}}{2}} \\ &= -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= -\sqrt{\frac{2+\sqrt{3}}{4}} \\ &= -\frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

ANSWER:

$-\frac{\sqrt{2+\sqrt{3}}}{2}$

65. tan 67.5°

SOLUTION:

Notice that 67.5° is half of 135°. Therefore, apply the half-angle identity for tangent, noting that since 67.5° lies in Quadrant I, its tangent is positive.

$$\begin{aligned} \tan 67.5^\circ &= \tan \frac{135^\circ}{2} \\ &= \frac{1-\cos 135^\circ}{\sin 135^\circ} \\ &= \frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}} \\ &= \frac{2+\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{2}+2}{2} \\ &= \sqrt{2}+1 \end{aligned}$$

ANSWER:

$\sqrt{2}+1$

66. $\cos \frac{3\pi}{8}$

SOLUTION:

Notice that $\frac{3\pi}{8}$ is half of $\frac{3\pi}{4}$. Therefore, apply the half-angle identity for cosine, noting that since $\frac{3\pi}{8}$ lies in Quadrant I, its cosine is positive.

$$\cos\frac{3\pi}{8} = \cos\frac{\frac{3\pi}{4}}{2}$$
$$= \sqrt{\frac{1+\cos\frac{3\pi}{4}}{2}}$$
$$= \sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}}$$
$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

ANSWER:

$$\frac{\sqrt{2-\sqrt{2}}}{2}$$